AD NUMBER

AD886132

NEW LIMITATION CHANGE

TO

Approved for public release, distribution unlimited

FROM

Distribution authorized to U.S. Gov't. agencies only; Test and Evaluation; JUN 1971. Other requests shall be referred to Space and Missle Systems Organization, Norton AFB, CA 92059.

AUTHORITY

SAMSO ltr, 30 May 1979

THIS REPORT HAS BEEN DELIMITED
AND CLEARED FOR PUBLIC RELEASE
UNDER DOD DIRECTIVE 5200.20 AND
NO RESTRICTIONS ARE IMPOSED UPON
ITS USE AND DISCLOSURE,

DISTRIBUTION STATEMENT A

APPROVED FOR PUBLIC RELEASE;
DISTRIBUTION UNLIMITED,

AD886132

IMR-1333*

INTERNAL MEMORANDUM

RELEASABLE

ERRORS INHERENT IN CHAFF CENTROID TRACKING

by

C. R. Mullin

July 1970

GENERAL RESEARCH CORPORATION
P.O. BOX 3587, SANTA BARBARA, CALIFORNIA 93105

DISTRIBUTION STATEMENT "B"
Distribution limited to U.S. Gov't agencies only; (TEST AND EVALUATION 28 JUNE 71) Other requests for this document must be referred to HQ SAMSO/RNSA NORTON AFB CA

38
att 2

Prepared for
Department of Defense Under
United States Air Force
Contract F04701-69-C-0124

~~This document may be further distributed by the holder only with specific prior approval of Space & Missile Systems Organization, Norton Air Force Base, California, ATTN: SMS~~

ACKNOWLEDGEMENTS

The author would like to thank J. Baltes for performing the trajectory calculations and V. Mrstik, D. Hunt, and K. Maitra for several helpful discussions on centroid motion.

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) General Research Corporation P.O. Box 3587, Santa Barbara, CA 93105	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

ERRORS INHERENT IN CHAFF CENTROID TRACKING

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Internal Memorandum Releasable

5. AUTHOR(S) (First name, middle initial, last name)

C.R. Mullin'

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
July 1970	38	-

8a. CONTRACT OR GRANT NO. F04701-69-C-0124	9a. ORIGINATOR'S REPORT NUMBER(S) IMR-1333
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	SAMSO-TR-71-175

10. DISTRIBUTION STATEMENT

Statement B: Distribution limited to U.S. Giv't agencies only; (test & evaluation, 28 June 1971). Other requests for this document must be referred to HQ SAMSO/RNSA, Norton AFB, CA 92049

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Space & Missile Systems Orgn. Norton AFB, CA 92049

13. ABSTRACT

An analysis of the errors inherent in tracking the radar-cross-section controid of a chaff cloud shows the centroid to have a random motion in addition to its long-term motion with the chaff cloud. This random motion can lead to errors in cloud trajectory estimation. There is a further error caused by the fact that the centroid does not exactly follow a Keplerian orbit. The deviation is slight, however, and can be neglected.

DD FORM 1 NOV 65 1473

Unclassified
Security Classification

Unclassified

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Chaff Tracking						
Trajectory Estimation						
Centroid Tracking						

Security Classification

ABSTRACT

An analysis of the errors inherent in tracking the radar-cross-section centroid of a chaff cloud shows the centroid to have a random motion in addition to its long-term motion with the chaff cloud. This random motion can lead to errors in cloud trajectory estimation. There is a further error caused by the fact that the centroid does not exactly follow a Keplerian orbit. The deviation is slight, however, and can be neglected.

CONTENTS

ILLUSTRATIONS

I. INTRODUCTION

In discussions of chaff cloud tracking it is often tacitly assumed that the measured radar-cross-section (RCS) centroid follows a Keplerian orbit. This is not entirely true for two reasons:

1. RCS fluctuations cause the instantaneous cross-section centroid to move randomly about the long-term-average centroid. This means that between any two independent measurements there will be random fluctuations in the centroid position, even with a noiseless radar.

2. For a large cloud there are various nonlinear gravity gradient effects. For typical trajectories the long-term-average centroid of a cloud will drift from a Keplerian particle having the same initial velocity and position. This displacement distance is quite small, however, being on the order of 0.1% of the cloud radius even for rather large clouds.

Each of these effects is discussed in subsequent subsections.

II. CENTROID MOTION

A. ANALYSIS

Consider a cloud that is N range resolution cells long but small in angular extent. The centroid is defined as

$$c = \sum_{i=1}^{N} i\sigma_i \Bigg/ \sum_{i=1}^{N} \sigma_i \equiv y/z \qquad (1)$$

where σ_i is the RCS of the $\underline{i}$th cell.

If all of the $\sigma(i)$ were equal and <u>constant in time</u>, then we would have

$$c = \frac{N(N+1)/2}{N} = \frac{N+1}{2}$$

Thus if there were 25 resolution cells the centroid would be in the center one; that is, $(25+1)/2 = 13$.

Because of the nature of chaff responses, however, the RCS will not be constant from cell to cell but will fluctuate because of interference effects. If the number of chaff pieces in any cell is sufficiently large the probability density function $p_1(\sigma)$ for the distribution of independent measurements of one cell will be exponential; that is

$$p_1(\sigma) = \frac{1}{\overline{\sigma}} e^{-\sigma/\overline{\sigma}}$$

We shall for the time being consider the case in which the expected value of $\overline{\sigma}$ is the same for each cell.

The centroid of the first N cells can be written

$$c(N) = \frac{\sum_{i=1}^{N} i\sigma_i}{\sum_{i=1}^{N} \sigma_i} = \frac{N\sigma_N + \sum_{i=1}^{N-1} i\sigma_i}{\sum_{i=1}^{N} \sigma_i}$$

which can be rearranged to give

$$c(N)\left(\sum_{i=1}^{N} \sigma_i\right) = N\sigma_N + c(N-1)\left(\sum_{i=1}^{N-1} \sigma_i\right)$$

If we let $g_N \equiv c(N)\left(\sum_{i=1}^{N} \sigma_i\right)$ then we have the contiguity relationship

$$g_N = g_{N-1} + N\sigma_N$$

The probability of having a value g_N is

$$p(g_N) = \int_0^{g_N} \frac{1}{N} p(g_{N-1}) p_1\left(\sigma_N = \frac{g_N - g_{N-1}}{N}\right) dg_{N-1}$$

Taking Laplace transforms we have

$$P_N(s) = P_{N-1}(s) \frac{1}{1 + Ns\bar{\sigma}}$$

where we have used the fact that

$$p_1(\sigma) = \frac{1}{\bar{\sigma}} e^{-\sigma/\bar{\sigma}}$$

Similarly

$$P_{N-1}(s) = P_{N-2}(s) \cdot \frac{1}{1 + (N-1)s\bar{\sigma}}$$

and thus

$$P_N(s) = \prod_{i=1}^{N} (1 + is\bar{\sigma})^{-1}$$

and

$$\ln P_N(s) = -\sum_{i=1}^{N} \ln(1 + is\bar{\sigma})$$

But

$$\sum_{i=1}^{N} \ln(1 + is\bar{\sigma}) = \left[s\bar{\sigma} - \frac{(s\bar{\sigma})^2}{2} + \frac{(s\bar{\sigma})^3}{3} - \dots \right]$$

$$+ \left[(2s\bar{\sigma}) - \frac{(2s\bar{\sigma})^2}{2} + \frac{(2s\bar{\sigma})^3}{3} - \dots \right]$$

$$\vdots$$

$$+ \left[(Ns\bar{\sigma}) - \frac{(Ns\bar{\sigma})^2}{2} + \frac{(Ns\bar{\sigma})^3}{3} - \dots \right]$$

$$= (s\bar{\sigma})[1 + 2 + 3 \cdots N] - \frac{(s\bar{\sigma})^2}{2}[1^2 + 2^2 + 3^2 \cdots N^2]$$

$$+ \frac{(s\bar{\sigma})^3}{3}[1^3 + 2^3 + 3^3 \cdots N^3] - \dots$$

$$= As + Bs^2 + Cs^3 \dots \equiv H(s)$$

where $A = \bar{\sigma}N(N+1)/2$

$$B = \frac{-\bar{\sigma}^2}{2}\left[N(N+1)\ \frac{(2N+1)}{6}\right]$$

$$C = \frac{\bar{\sigma}^3}{3}\ [N(N+1)/2]^2$$

etc.

Thus

$$p(g_N) = \mathcal{L}^{-1}P_N(s) = \mathcal{L}^{-1}\ e^{-H(s)}$$

Since $g_N = (c_N) \sum_{i-1}^{N} \sigma_i$

we can write

$$p(g) = \int_0^\infty p(c)p_N(z = g/c)\ \frac{1}{c}\ dc \tag{2}$$

where we have suppressed the subscript on g and c, and let

$$\sum_{i=1}^{N} \sigma_i \equiv z$$

Before going on, it is worthwhile pointing out explicitly that in Eq. 2 we have tacitly assumed that $p(c)$ was independent of z. Consider two sets of σ's, $\sigma_1, \sigma_2, \ldots \sigma_N$ and $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3, \ldots \bar{\sigma}_N$, each having the same total σ (i.e., $\sigma_1 + \sigma_2 \ldots \sigma_N = \bar{\sigma}_1 + \bar{\sigma}_2 \ldots \bar{\sigma}_N = z$). Each set has a unique centroid (c and $\bar{c}$) associated with it. The relative likeliness of each set (point in N dimensional space) is

$$\frac{p(\sigma_1)p(\sigma_2)\ldots p(\sigma_N)\ d\sigma_1\ d\sigma_2\ldots d\sigma_N}{p(\bar{\sigma}_1)p(\bar{\sigma}_2)\ldots p(\bar{\sigma}_N)\ d\sigma_1\ d\sigma_2\ldots d\sigma_N}$$

Now consider two other sets of points, $\alpha\sigma_1, \alpha\sigma_2, \ldots \alpha\sigma_N$ and $\alpha\bar{\sigma}_1, \alpha\bar{\sigma}_2, \ldots \alpha\bar{\sigma}_N$. These have the same centroids (c and $\bar{c}$) associated with them. If the relative likeliness of these two new sets of points is the same as for the first two sets ($\alpha = 1$), then the density distribution of c's will be invariant with α . Since varying α varies the total RCS (z), we will have proved that $p(c)$ is independent of z .

Thus we must show that

$$\frac{p(\sigma_1)\ldots p(\sigma_N)\ d\sigma_1\ldots d\sigma_N}{p(\bar{\sigma}_1)\ldots p(\bar{\sigma}_N)\ d\bar{\sigma}_1\ldots d\bar{\sigma}_N} \stackrel{?}{=} \frac{p(\alpha\sigma_1)\ldots p(\alpha\sigma_N)\ d\alpha\sigma_1\ldots d\alpha\sigma_N}{p(\alpha\bar{\sigma}_1)\ldots p(\alpha\bar{\sigma}_N)\ d(\alpha\bar{\sigma}_1)\ldots d(\alpha\bar{\sigma}_N)}$$

Canceling the $d\sigma$'s and the α's gives

$$\frac{p(\sigma_1)\ldots p(\sigma_N)}{p(\bar{\sigma}_1)\ldots p(\bar{\sigma}_N)} \stackrel{?}{=} \frac{p(\alpha\sigma_1)\ldots p(\alpha\sigma_N)}{p(\alpha\bar{\sigma}_1)\ldots p(\alpha\bar{\sigma}_N)}$$

For an exponential distribution this gives

$$\frac{e^{-z}}{e^{-z}} \stackrel{?}{=} \frac{e^{-\alpha z}}{e^{-\alpha z}}$$

and the equality has been shown.

The expected distribution of $\sum_{i=1}^{N} \sigma_i$, i.e., $p_N(z)$, can be evaluated by noting that

$$p_N(z) = \int_0^z p_{N-1}(\sigma')p_1(z - \sigma')\ d\sigma'$$

The N indicates a summation over N cells. Taking Laplace transforms we obtain

$$P_N(s) = [P_{N-1}(s)]P_1(s)$$

$$= [P_{N-2}(s)P_1(s)]P_1(s)$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$= [P_1(s)]^N$$

But $P_1(s) = \mathcal{L}\,\frac{1}{\bar{\sigma}}\, e^{-\sigma/\bar{\sigma}}$

$$= \frac{1}{1 + \bar{\sigma}s}$$

Thus

$$P_N(z) = \mathcal{L}^{-1}\,\frac{1}{(1 + \bar{\sigma}s)^N} = \frac{(z/\bar{\sigma})^{N-1}}{\bar{\sigma}\Gamma(N)}\, e^{-z/\bar{\sigma}}$$

If we multiply both sides of Eq. 2 by g^q and integrate from zero to infinity, we have

$$\int_0^\infty c^q\ p(c)\ dq = \frac{\Gamma N}{\bar{\sigma}^q\ \Gamma(N+q)} \int_0^\infty g^q\ p(g)\ dg$$

If we let $q = 1$, then we have

$$\int_0^\infty cp(c)\ dc = \frac{1}{N\bar{\sigma}} \int_0^\infty gp(g)\ dg$$

The integral on the left is $\bar{c}$ (the expected value of the centroid). The integral on the right is the first moment of $p(g)$ and this is simply*

$$\left(-\frac{\partial}{\partial s}\right) P_N(s)\Bigg|_{s=0} = \left(-\frac{\partial}{\partial s}\right) e^{-H(s)}\Bigg|_{s=0}$$

$$= A$$

Thus

$$\bar{c} = A/N\bar{\sigma} = \frac{N+1}{2}$$

Similarly, if we take $q = 2$, we have

$$\int_0^\infty c^2 p(c)\ dc = \frac{\Gamma(N)}{\bar{\sigma}^2 (N+1)!} \int_0^\infty g^2 p(g)\ dg$$

---

* Remember that if $F(s) \equiv \int_0^\infty e^{-sx} f(x)\ dx$, then differentiating both sides N times with respect to s gives

$$\left(\frac{\partial}{\partial s}\right)^N F(s) = (-1)^N \int_0^\infty f(x) x^N e^{-sx}\ dx$$

If we now set $s = 0$, we have

$$\int_0^\infty x^N f(x)\ dx = (-1)^N \left(\frac{\partial}{\partial s}\right)^N F(s)\Bigg|_{s=0}$$

Thus the Nth moment of a function is $(-1)^N$ times the Nth derivation of its transform at $s = 0$.

$$\overline{c^2} = \frac{1}{\bar{\sigma}^2 (N+1) N} (A^2 - 2B)$$

The variance is

$$\overline{c^2} - \bar{c}^2 = \frac{A^2 - 2B}{\bar{\sigma}^2 N(N+1)} - \frac{A^2}{\bar{\sigma}^2 N^2}$$

$$= \frac{N-1}{12}$$

In order to verify this result a computer Monte Carlo experiment was run. The quantity

$$c = \frac{\sum_{i=1}^{N} i\sigma_i}{\sum_{i=1}^{N} \sigma_i}$$

was calculated after generating N random values of σ_i . The σ_i were selected from a distribution of the form

$$p(\sigma) = e^{-\sigma}$$

The process was then repeated W times to give a total of W independent values of c .

A total of $W = 500$ independent evaluations of c , for a cloud containing 121 cells, gave a value of $\bar{c} = 61.1$, versus a theoretical value of 61, i.e., $(121 + 1)/2$. The quantity $\overline{c^2} - \bar{c}^2$ had a value of 10.1 compared with the theoretical value of 10.0 (i.e., $\frac{121 - 1}{12}$) .

B. INTERPRETATION

Since the variance is proportional to N (for large N), the rms fluctuation* (standard deviation) on any measurement is proportional to $\sqrt{N}$. If the cloud has a total length L and a resolution cell width $\delta = L/N$, we see that on any measurement the instantaneous "true" centroid is typically

$$\ell = \text{standard deviation} \times \text{cell size}$$

$$= \sqrt{\frac{N-1}{12}} \times \delta \approx \sqrt{\frac{L/\delta}{12}} \times \delta = \sqrt{\frac{L\delta}{12}} \qquad (3)$$

from the long-term-average centroid.

This result is physically reasonable in that the fluctuation gets smaller as

1. The cloud gets shorter
2. The resolution gets better**

The "square root of N" form for the deviation from center is exactly what one would expect from a random walk configuration and could probably have been predicted on a priori grounds; however, the coefficient of 1/12 could not have been similarly predicted.

---

*It would be slightly misleading to refer to this term as "error" since it is the true instantaneous RCS centroid. The point is that the RCS centroid fluctuates. The most one can say is that there is some long-term-average centroid which can be used as a reference point. This point is not necessarily coincident with the center of mass. Strictly speaking one can argue that even this long-term-average has a secular variation because of changing aspect angle of the radar line-of-sight relative to the chaff spin axes. This effect is small.

**Remember we have assumed there are a number of chaff pieces in each cell (to assume an exponential distribution of RCS values) and also that $L \gg \delta$ ($N \gg 1$). These two assumptions impose limits on where the result can be used.

This brings up the interesting point as to whether there might be an alternate partitioning scheme which would have smaller fluctuations associated with it. We have been looking for the centroid which is a quantity defined by*

$$\int_{-\infty}^{\infty} dx\ (x - c)\ \sigma(x)\ dx = 0 \qquad (4)$$

It is also possible to track relative to the median; one looks for the point that has equal amounts of RCS on either side.

Mathematically we look for a point m such that

$$\int_{-\infty}^{m} \sigma(x)\ dx = \int_{m}^{\infty} \sigma(x)\ dx \qquad (5)$$

or, put into the same format as Eq. 4

$$\int_{-\infty}^{\infty} \frac{x - m}{|x - m|}\ \sigma(x)\ dx = 0 \qquad (6)$$

The difference between the two quantities is that the calculation of c places more weight on pieces of chaff far removed from the center (the weighting factor has a magnitude $|x - c|$). The definition of m places equal weight on all pieces of chaff since the weighting factor in Eq. 6 always has unit magnitude.

The Appendix gives a derivation of the rms fluctuation when the median is tracked and it is

$$\ell = \sqrt{\frac{N}{4}} \times \delta = \frac{1}{2} \sqrt{L\delta}$$

*For generality we shall use an integration notation; it is, mutatis mutandis, consistent with Eq. 1.

Note that the form $(\sqrt{L\delta})$ is the same as for centroid tracking although the coefficient is slightly larger.

C. GENERALIZATIONS TO OTHER GEOMETRIES

1. Non-Uniform Density

Throughout this entire discussion we have assumed that the expected chaff distribution is uniform. If it is not uniform but (as is more likely) peaked in the center, the variance is be less.

A chaff density distribution as is shown in Fig. 1 would have a variance of approximately $\sqrt{1/12}$ $\sqrt{L/2}$ rather than $\sqrt{1/12}$ $\sqrt{L}$. (The wings are assumed to be negligible compared to the center.)

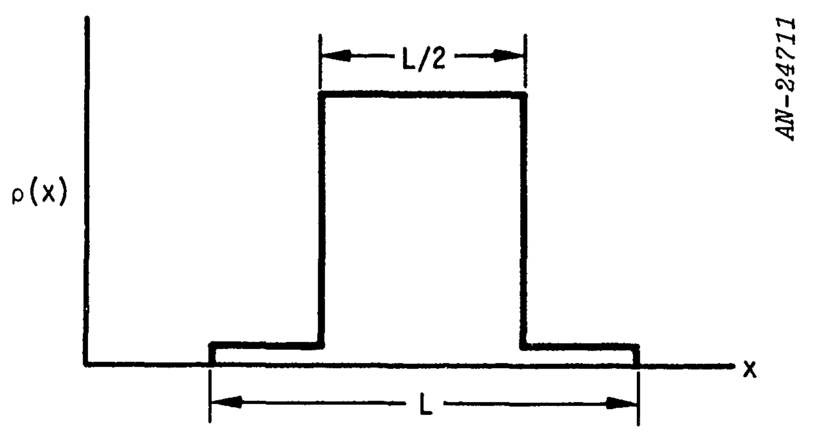

Figure 1. Chaff Density Versus Range

If the density distribution were as shown in Fig. 2, the centroid variance could be considered as from two contributors, one of RCS $\rho_o L_2$ and variance $G(L_2)$ and the other RCS $\rho_o L_1$ and variance $G(L_1)$.

$G(L)$ is the Green function for the variance of a uniform density distribution of length L and resolution length δ . Thus $G(L) = (1/12)\delta L$ for centroid mapping and $(3/12)\delta L$ for median mapping.

The resultant net variance for the two is

$$\sigma^2 = \frac{\rho_o L_2 G(L_2) + \rho_o L_1 G(\ \ )}{\rho_o L_2 + \rho_o L_1}$$

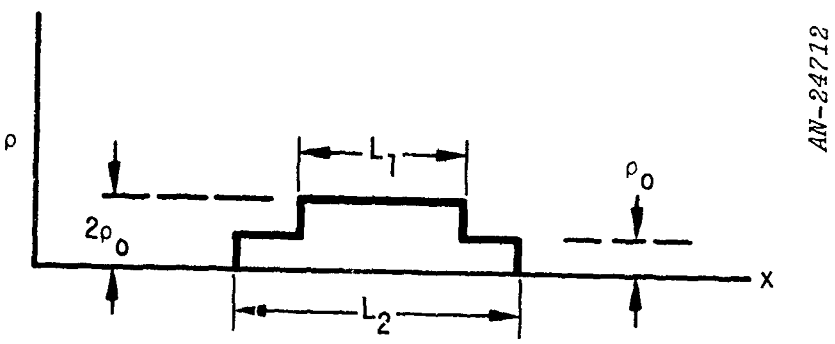

Figure 2. Chaff Density Versus Range

This result can be generalized for a density distribution which is symmetric and monotonic on each side (as shown in Fig. 3) to give

$$\sigma^2_{\text{average}} = 2 \int_0^\infty dx \times \frac{d\rho}{dx} G(x) \Bigg/ \int_0^\infty dx \times \frac{d\rho}{dx}$$

where we used the fact that the density distribution is symmetric to place the lower limit at zero.

A uniform density distribution of total length L would give

$$\frac{d\rho}{dx} = \rho_o [\delta(x + L/2) - \delta(x - L/2)]$$

and

$$\sigma^2 = 2g \frac{-(L/2)^2}{-(L/2)} = gL\delta$$

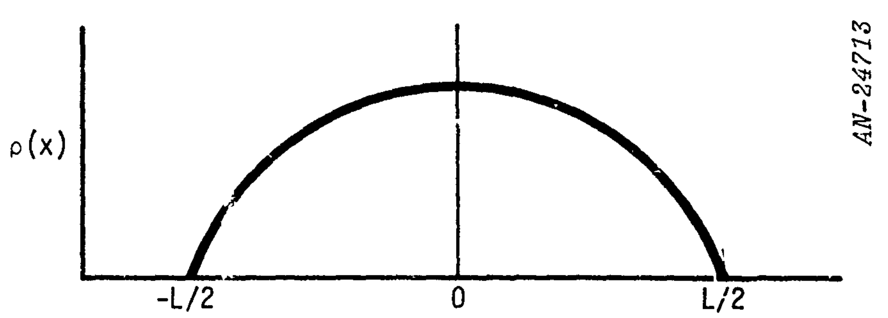

Figure 3. Chaff Density Versus Range

where $g = 1/12$ for centroid mapping and 3/12 for median mapping. This result is the same as that obtained before.

A triangular distribution would give

$$\rho = \rho_o \left(1 - \left|\frac{x}{L/2}\right|\right) ,$$

$$\frac{d\rho}{dx} = - \frac{\rho_o L}{L} \qquad (x > 0), \text{ and}$$

$$\sigma^2 = \frac{2g \left(\frac{-2\rho_o}{L}\right) \frac{(L/2)^3}{3}}{\left(\frac{-2\rho_o}{L}\right) \frac{(L/2)^2}{2}} = \frac{2}{3}\, gL\delta$$

Similarly a quadratic distribution would give

$$\rho = \rho_o \left(1 - \left(\frac{x}{L/2}\right)^2\right)$$

$$\sigma^2 = \frac{3}{4}\, gL\delta$$

All of the preceding results can be summarized by saying that the rms shift of any measurement from its long term average is

$$\left(\sqrt{gf}\right) \sqrt{L\delta}$$

where $g = 1/12$ for centroid mapping and 3/12 for median mapping; f is a form factor which is 1 for a uniform chaff distribution, 3/4 for a quadratic distribution and 2/3 for a triangular one. Thus for example, the centroid of a chaff cloud of length L with a quadratic distribution of chaff will fluctuate about its long-term position with a typical deviation of $\sqrt{(1/12)(3/4)}\ \sqrt{L\delta} = 0.25\ \sqrt{L\delta}$.

2. Three-Dimensional Configurations

The discussion throughout this paper has been restricted solely to finding the variance of the RCS centroid (or median) of a cloud that was long in one dimension but short in the other two. (We have considered the long dimension to be range, but it could just as well have been one of the angular dimensions.)

It is worthwhile generalizing our result to the case in which the object could have subdivisions in angle as well as range.

Consider a rectangular cloud (Fig. 4) of dimensions L_1, L_2 and L_3, which can be radar subdivided into cells of size Δ_1, Δ_2, and Δ_3 (L_1 could be the range extent of the cloud and Δ_1 the range resolution; L_2 could be the width of the cloud in one dimension, and Δ_2 the associated angular resolution).

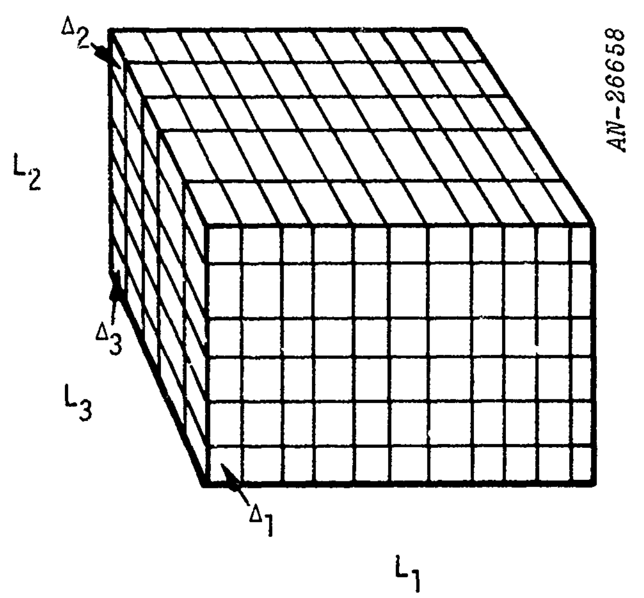

Figure 4. Graphic Representation of Chaff Cloud

There are thus

$$\frac{L_2}{\Delta_2} \times \frac{L_3}{\Delta_3} \equiv N_1$$

columns of length L_1 , and each one has a centroid standard deviation of

$$\sqrt{\frac{1}{12}}\ \sqrt{L_1\ \Delta_1}$$

If we average the centroids of each of the N_1 columns, the standard deviation will be reduced by a factor $1/\sqrt{N_1}$, to give a value

$$\sqrt{\frac{1}{12}}\ \sqrt{\left(\frac{L_1\ \Delta_1}{N_1}\right)}$$

The standard deviation in the other dimensions can be written similarly, i.e.,

$$\sqrt{\frac{1}{12}} \cdot \sqrt{\frac{L_2\Delta_2}{N_2}} \quad \text{and} \quad \sqrt{\frac{1}{12}} \cdot \sqrt{\frac{L_3\Delta_3}{N_3}}$$

The standard deviation of the net (three-dimensional) "error" is the square root of the sum of the square of the standard deviation in each dimension. Thus we have a net standard deviation of

$$\sqrt{\frac{1}{12}\left(\frac{L_1\Delta_1}{N_1} + \frac{L_2\Delta_2}{N_2} + \frac{L_3\Delta_3}{N_3}\right)} = \sqrt{\frac{1}{12}}\sqrt{\frac{\Delta_1\Delta_2\Delta_3}{L_1L_2L_3}}\left(L_1^2 + L_2^2 + L_3^2\right)^{1/2}$$

If we re-express the cloud in terms of two form factors $\alpha_2 \equiv L_2/L_1$ and $\alpha_3 = L_3/L_1$, and one scale size L_1 , then we have

$$\text{standard deviation} = \frac{1}{\sqrt{L_1}} \sqrt{1/12} \sqrt{\Delta_1\Delta_2\Delta_3} \left[\frac{1+\alpha_2^2 + \alpha_3^2}{\alpha_2\alpha_3}\right]^{1/2}$$

Note that for a given shape cloud (α_2 and α_3 constant) <u>the standard deviation gets smaller as the cloud gets larger</u>. This is contradistinction to the case of a <u>one</u>-dimensional cloud where the standard deviation gets <u>larger</u> as the cloud gets longer. The reason is that the variance in any one dimension is proportional to the length in that dimension divided by the <u>number</u> of columns in that dimension, e.g.,

$$\frac{L_1}{N_1} = \frac{L_1}{L_2L_3} \cdot (\Delta_2\Delta_3)$$

If all dimensions are doubled then the variance of one column is doubled (because it is twice as long): but there are now four times as many columns for a net improvement of a factor of two (a factor of the square root of two in standard deviation).

This analysis has several limits which should be spelled out. In the first place, we have tacitly assumed no errors introduced by S/N considerations, i.e., the signal is so strong that this error is negligible compared to variations in centroid caused by the RCS fluctuations.

As the cloud gets larger this assumption will become poorer for two reasons:

1. For a constant amount of chaff (constant total average RCS) the signal in each cell becomes smaller as the chaff is diluted, and thus the signal noise error <u>increases</u>

and

2. The centroid fluctuation "error" decreases ($\sim 1/\sqrt{L}$).

Thus at some size S/N considerations will dominate.

There is a further limit. If we consider the chaff cloud to be compressed to a point (still maintaining a constant total RCS) then very accurate range measurements are possible because of the large S/N ratio. The error will be proportional to

$$\frac{\text{resolution length}}{\sqrt{S/N}}$$

and will not have a centroid RCS fluctuation term.

The competing effects can be illustrated as shown in Fig. 5, where we have considered a cubical cloud of side L. For small L (much less than a resolution length and even less than resolution length/$(S/N)^{1/2}$ the centroid standard deviation comes solely from S/N and radar considerations and not from true motion of the centroid (Region I).

As L gets larger (Region II) centroid motion comes into dominance and the standard deviation becomes larger. After L becomes larger than a resolution cell and the radar has multiple cells, the centroid motion error becomes smaller (at a rate proportional $1/\sqrt{L}$) (Region III).

Finally (Region IV) the standard deviation caused by the centroid motion becomes less than the increasing noise error (because of decreased S/N in each cell). D. Hunt has shown* that in this region the standard deviation of the centroid error is independent of cloud size, as long as the S/N ratio is large enough to permit detection.

*D. Hunt, Chaff Cloud Centroid Measurements with the MSR and ALOR Radar Systems, General Research Corporation IMR-1377, August 21, 1970.

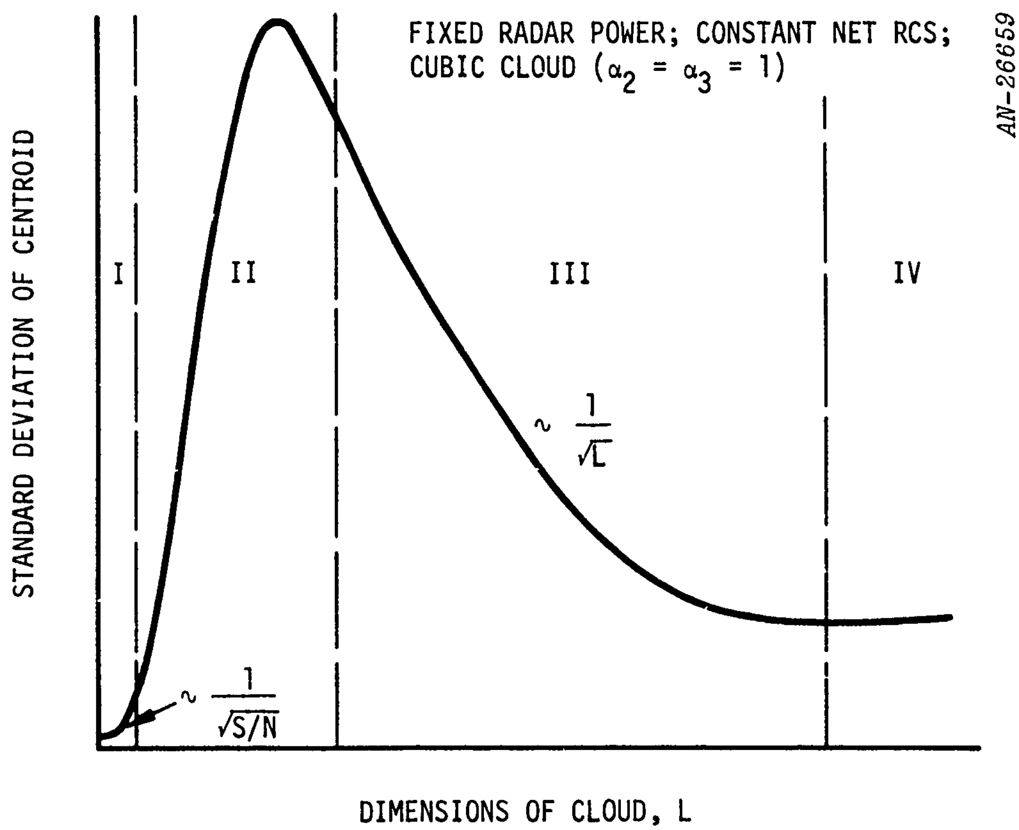

Figure 5. Representative Curve Showing Centroid Error as a Function of Cloud Size

III. GRAVITY GRADIENT EFFECTS

In discussions of chaff tracking it is usually assumed that the (long-term-average) centroid follows a Keplerian orbit, although it is known that this is not rigorously true. To quantify the magnitude of the error in such an assumption, several machine runs were performed. The first was a 4500 n mi reference trajectory 30 deg reentry angle. Then the trajectories of two particles ejected 63 s after launch and in opposite directions to each other, were calculated. The center point of the two ejected particles can be considered as the centroid of a rather degenerate (2 piece) chaff cloud.* The deviation of this centroid from the reference trajectory is a measure of its deviation from the Kepler orbit since both trajectories had the same position and velocity at ejection. (One point in position-velocity space is sufficient to define a Keplerian trajectory.)

Figure 6 shows the separation between the centroid and the true Keplerian position for four ejection velocities (10, 50, 100, and 150 ft/s) as a function of time after ejection; the particles are ejected at right angles to the velocity vector and in the plane of the orbit. Note that the slope of the curve is approximately proportional to t^3 and thus the centroid has a velocity away from the reference trajectory at a rate proportional to t^2. The curves can be very closely approximated by

$$D = 5 \times 10^{-12} V_o^2 t^3 \text{ ft}$$

Figure 7 shows the fractional displacement of the centroid, that is, the displacement divided by the cloud diameter, as a function of diameter at impact. The fractional rms is quite small and even for a cloud as large as 100 n mi at reentry is less than 0.2%. This deviation is usually negligible for most applications.

---

*We are assuming here that each piece has the same RCS.

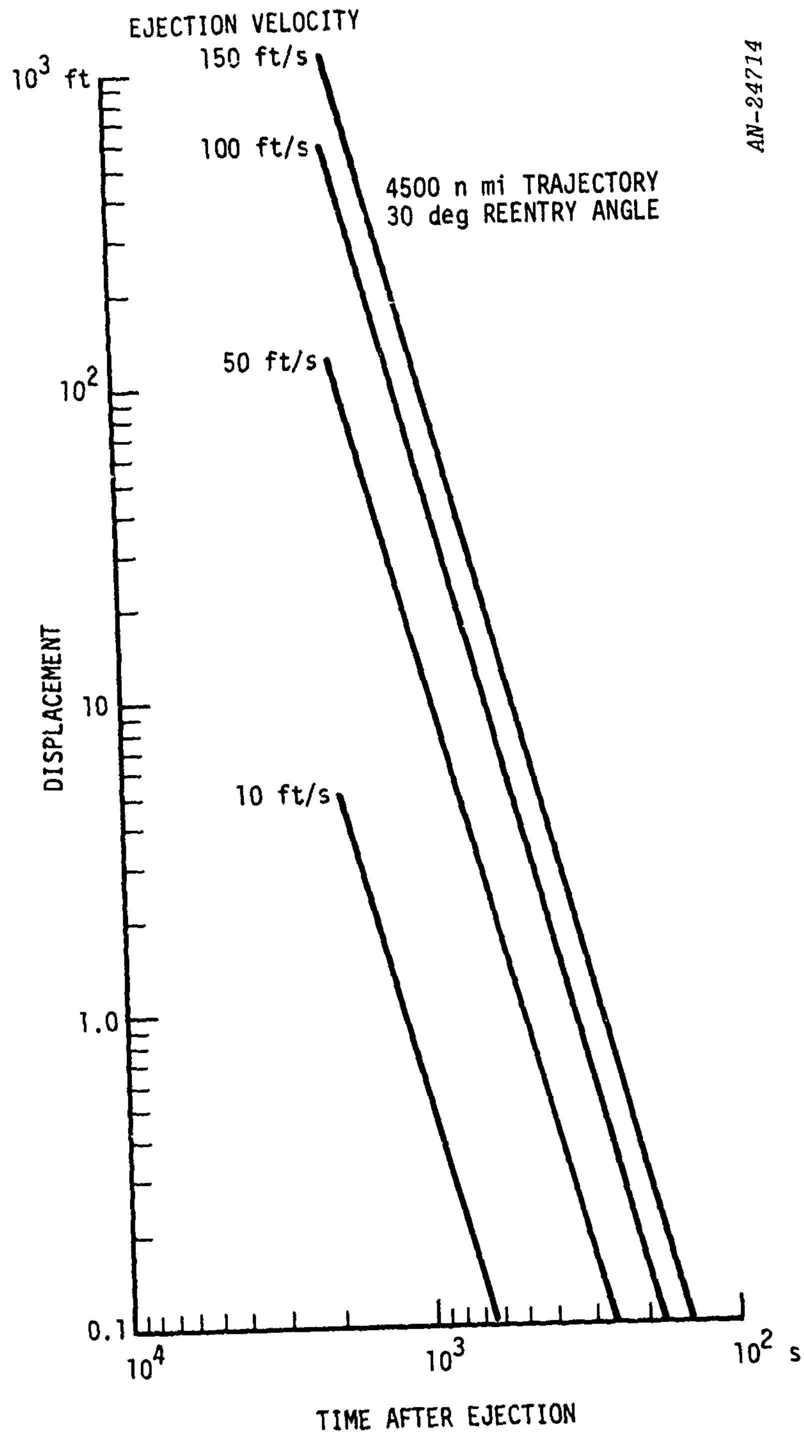

Figure 6. Centroid Displacement from the True Keplerian Position as a Function of Time after Ejection

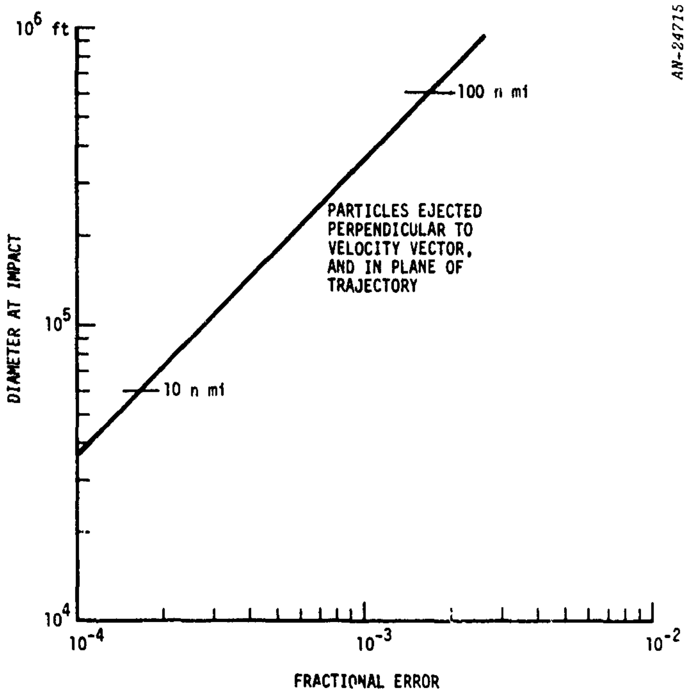

Figure 7. Fractional Displacement of Centroid as Function of Cloud Diameter at Impact: 30-deg reentry; 4500 n mi Trajectory

If the particles are ejected along different axes, the same general conclusions hold, although numerical values are slightly different (Fig. 8). The largest fractional shift of the centroid occurs when the particles are ejected perpendicular to the plane of the trajectory. Even in this case the fractional deviation is less than 1% even for a 100 n mi diameter cloud. This is the deviation for the special case in which all of the particles are at the edge of the cloud; any other weighting would reduce the values still further. A uniform distribution reduces the fractional error by a factor of 3. As an example, chaff ejected perpendicular to the plane of the trajectory (worst case) so that at reentry it formed a uniform line 40 n mi long would have a deviation of less than 0.1% of 40 n mi.

And finally a realistic case in which particles were ejected in several azimuths would give a further cancellation because of offsets in different directions.

In short, this gravity gradient effect is--for typical trajectories--negligible.

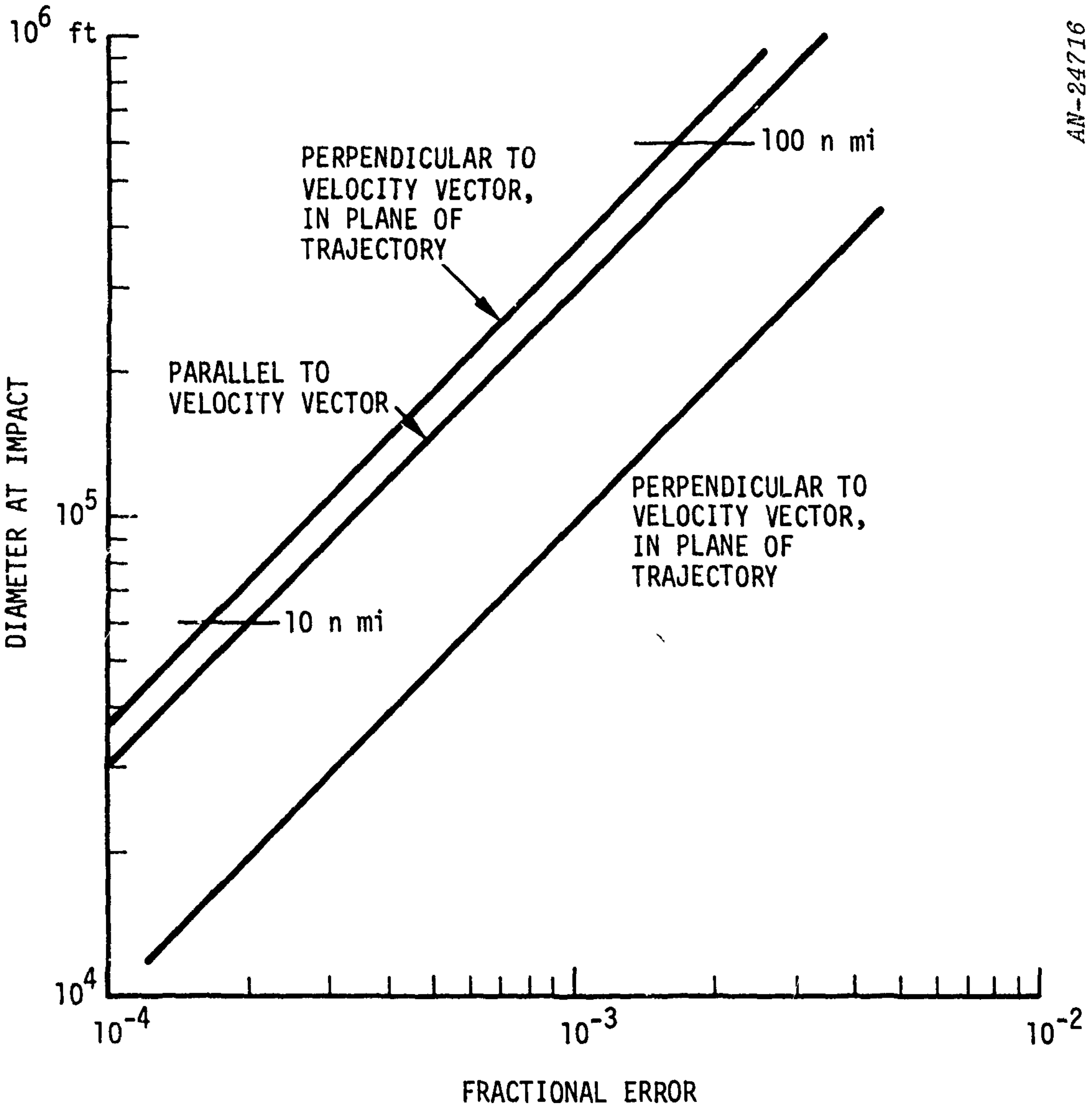

Figure 8. Fractional Displacement of Centroid as Function of Cloud Diameter at Impact: 30-deg Reentry; 4500 n mi Trajectory

IV. CONCLUSIONS

The major "error" in centroid tracking (over and above radar bias and S/N errors) arises from the fact that the centroid is not a steady quantity but rather fluctuates in space. The rms fluctuation from measurement to measurement is on the order of $0.3\sqrt{L\delta}$ in range.

The error caused by the drift of the centroid (or median) from a Keplerian orbit is small in comparison with this effect. (A 0.1% fractional shift in the centroid would be comparable to the centroid jitter effect only if $\sqrt{N}$ were on the order of 10^{-3}, and this would demand a resolution cell width one-millionth of the cloud length.)

APPENDIX A

In this appendix we shall estimate the variance of the median RCS position. The cloud is $N = L/\delta$ resolution cells long, where L is the cloud length and δ is resolution cell width. Each cell has the same average RCS $\bar{\sigma}$, and independent values are distributed exponentially.

If one sums the RCS values in the first ν cells the probability distribution for the sum is

$$p_1(\sigma_1,\nu) = \frac{(\sigma_1/\bar{\sigma})^{\nu-1}}{\Gamma(\nu)} \frac{1}{\bar{\sigma}} e^{-\sigma_1/\bar{\sigma}} \tag{7}$$

while the probability distribution for the sum of the remaining $N-\nu$ is

$$p_2(\sigma_2,\nu) = \frac{(\sigma_2/\bar{\sigma})^{N-(\nu-1)}}{\Gamma(N-\nu)} \frac{1}{\bar{\sigma}} e^{-\sigma_2/\bar{\sigma}} \tag{8}$$

If $z \equiv \sigma_1 - \sigma_2$,

then

$$p(z,\nu) = \int^{\infty} p_1(\sigma,\nu)\, p_2(\sigma-z,\nu)\, d\sigma$$

For $z = 0$ (equal RCS on each side), we have

$$p(0,\nu) = \int_0^\infty p_p(\sigma,\nu)\; p_2(\sigma,\nu)\; d\sigma$$

This gives the distribution of the cell number (ν) at which the RCS sums on each side are equal; we shall name the ν value which partitions the cloud into two equal parts m and write

$$p(0,\nu) \equiv p(m) = \int_0^\infty \frac{1}{\bar{\sigma}\Gamma(m)} (\sigma/\bar{\sigma})^{m-1} \frac{1}{\bar{\sigma}\Gamma(N-m)} (\sigma/\bar{\sigma})^{N-m-1}\, e^{-2\sigma/\bar{\sigma}}\; d\sigma$$

$$= \frac{1}{\bar{\sigma}} \frac{\Gamma(N-1)}{2^{N-1}\,\Gamma(m)\;\Gamma(N-m)}$$

For large N we can make the transition from a summation to an integral and write

$$\bar{m} \simeq \int_0^N dm\; m\; p(m) \Bigg/ \int_0^N p(m)\; dm$$

$$= \int_{-N/2}^{N/2} dm(m+N/2)\;\, p(m+N/2) \Bigg/ \int_{-N/2}^{N/2} dm\; p(m+N/2)$$

$$= \frac{\int_{-N/2}^{N/2} dm\;\, (m+N/2)\;\, \Gamma(m+N/2)\;\, \Gamma(N/2 - m)}{\int_{-N/2}^{N/2} dm\;\, \Gamma(m+N/2)\;\, \Gamma(N/2 - m)}$$

$$= \frac{N}{2} + \frac{\int_{-N/2}^{N/2} dm\; m\; \frac{1}{\Gamma(N/2 + m)\;\Gamma(N/2 - m)}}{\int_{-N/2}^{N/2} \frac{dm}{\Gamma(N/2 + m)\;\Gamma(N/2 - m)}}$$

The upper integral is 0 since the integrand is odd, and thus

$$\overline{m} \simeq \frac{N}{2} \qquad (N >> 1)$$

We can evaluate the second moment by writing

$$\overline{m^2} \simeq \int_0^N m^2 \, dm \, p(m) \Big/ \int_0^N dm \, p(m)$$

$$= (N/2)^2 + \frac{\int_{-N/2}^{N/2} m^2 \, p(m + N/2) \, dm}{\int_{-N/2}^{N/2} p(m + N/2) \, dm}$$

$$= (N/2)^2 + \int_{-N/2}^{N/2} m^2 \, p(m + N/2) \, dm \Big/ \int_{-N/2}^{N/2} p(m + N/2) \, dm$$

$$= (N/2)^2 + \int_{-N/2}^{N/2} \frac{m^2 \, dm}{\Gamma(N/2 - m) \, \Gamma(N/2 + m)} \Big/ \int_{-N/2}^{N/2} \frac{dm}{\Gamma(N/2 - m) \, \Gamma(N/2 + m)}$$

Using Sterling's approximation this becomes

$$(N/2)^2 + \frac{\int_0^{N/2} \frac{m^2 \, dm}{\left[1 - m^2/(N/2)^2\right]^{N/2-1/2}} (1 - 4m/N)^m}{\int_0^{N/2} \frac{dm \, (1 - 4m/N)^m}{\left[1 - m^2/(N/2)^2\right]^{(N-1)/2}}}$$

By making approximations

$$(1 - 4m/N)^m \simeq e^{-4m^2/N}$$

and

$$\left(1 - \frac{4m^2}{N^2}\right)^{N-1} \simeq e^{-2m^2/N}$$

and extending the limits of integration to infinity, we have

$$\overline{m^2} = (N/2)^2 + \frac{\int_0^\infty m^2 \, e^{-2m^2/N}}{\int_0^\infty e^{-2m^2/N}} = (N/2)^2 + \frac{N}{4}$$

The variance is

$$\overline{m^2} - (\overline{m})^2 = (N/2)^2 + N/4 - (N/2)^2 = N/4$$

and the RMS shift in the median of any measurement from its long-term average is $\sqrt{1/4N} = \frac{1}{2}\sqrt{N}$.